P76 - 77

1. (a) $u_x \equiv 3 \quad v_x = -1$

$u_y \equiv 1 \quad v_y = 3$

By Cauchy-Riemann equation, f is entire.

(b) $u_x = \sinh x \cos y \quad v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y \quad v_y = \sinh x \cos y$

By Cauchy-Riemann equation, f is entire.

(c) $u_x = e^{-y} \cos x \quad v_x = e^{-y} \sin x$

$u_y = -e^{-y} \sin x \quad v_y = e^{-y} \cos x$

By Cauchy-Riemann equation, f is entire.

(d) $f(z) = (z^2-2) e^{-x} e^{-iy}$

Let $g(x) = e^{-x} e^{-iy} = e^{-x} \cos y - i e^{-x} \sin y$

$:= u(x,y) + i\, v(x,y)$.

Clearly, z^2-2 is entire. Therefore, if g is entire, then f is entire.

$u_x = -e^{-x} \cos y \quad v_x = e^{-x} \sin y$

$u_y = -e^{-x} \sin y \quad v_y = -e^{-x} \cos y$

By Cauchy-Riemann equation, g is entire.

Hence, f is entire.

$\square$

2 (a) $u_x = y \quad v_x = 0$

$u_y = x \quad v_y = 1$

By Cauchy-Riemann equation, f is not differentiable unless $z = i$. Hence, f is nowhere analytic.

(b) $u_x = 2y \quad v_x = 2x$

$u_y = 2x \quad v_y = -2y$

By Cauchy-Riemann equation, f is not differentiable unless $z = 0$. Hence, f is nowhere analytic.

(c) $f(z) = e^y e^{ix} = e^y \cos x + i e^y \sin x$

$u_x = -e^y \sin x \quad v_x = e^y \cos x$

$u_y = e^y \cos x \quad v_y = e^y \sin x$

By Cauchy-Riemann equation, f is nowhere differentiable. Hence, f is nowhere analytic.

$\square$

7. Let $f(z) = u(x,y) + iv(x,y)$.

Since f is real-valued, $v \equiv 0$.

Since f is entire, $\begin{cases} u_x = v_y \equiv 0 \\ u_y = -v_x \equiv 0 \end{cases}$ by C-R equation.

Therefore, $u \equiv$ constant.

Hence, $f \equiv$ constant. $\square$

P 89 - 90

1. (a) $\exp(2 \pm 3\pi i) = e^2 \cdot e^{\pm 3\pi i}$

$= e^2 \cdot e^{\pi i}$

$= e^2 \cdot (-1)$

$= -e^2$

(b) $\exp\left(\frac{2+\pi i}{4}\right) = e^{\frac{1}{2}} \cdot e^{\frac{\pi}{4}i}$

$= e^{\frac{1}{2}} \cdot \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$

$= \sqrt{\frac{e}{2}}\,(1+i)$

(c) $\exp(z+\pi i) = e^z \cdot e^{\pi i}$

$= e^z \cdot (-1)$

$= -e^z$.

$\square$

3. $f(z) = \exp \bar{z} = \exp(x - iy) = e^x \cos y - i e^x \sin y$.

$u_x = e^x \cos y \qquad v_x = -e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = -e^x \cos y$

By Cauchy-Riemann equation, f is nowhere differentiable. Hence, f is nowhere analytic.

$\square$

10 (a) $e^z = e^{x+iy} = e^x\cos y + ie^x\sin y$.

Since e^z is real, $e^x\sin y = 0$.

Then $\sin y = 0$. Hence, $y = n\pi$ where $n \in \mathbb{Z}$.

(b) Since e^z is pure imaginary, $e^x\cos y = 0$.

Then $\cos y = 0$. Hence, $y = \frac{\pi}{2} + n\pi$ where $n \in \mathbb{Z}$.

$\square$

13. Let $g(z) = U(x,y) + iV(x,y)$

$$= e^{u(x,y)}(\cos v(x,y) + i\sin v(x,y))$$

$$= e^{u(x,y)+iv(x,y)}$$

$$= e^{f(z)}$$

Since f is analytic, so is g.

Then U and V are both harmonic.

$\square$

P95 - 96

1. (a) $\text{Log}(-ei) = \text{Log}(e\cdot(-i))$

$$= \text{Log}(e\cdot e^{-\frac{\pi}{2}i})$$

$$= \text{Log}(e^{1-\frac{\pi}{2}i})$$

$$= 1 - \frac{\pi}{2}i$$

(b) $\operatorname{Log}(1-i) = \operatorname{Log}\left(\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)$

$= \operatorname{Log}\left(2^{\frac{1}{2}} e^{-\frac{\pi}{4}i}\right)$

$= \ln 2^{\frac{1}{2}} - \frac{\pi}{4}i$

$= \frac{1}{2}\ln 2 - \frac{\pi}{4}i$

$\square$

5 (a) $\left(e^{\frac{\pi}{4}i}\right)^2 = e^{\frac{\pi}{2}i} = i$

$\left(e^{\frac{5}{4}\pi i}\right)^2 = e^{\frac{5}{2}\pi i} = e^{\frac{\pi}{2}i} = i$

$\log\left(e^{\frac{\pi}{4}i}\right) = \log\left(e^{\frac{\pi}{4}i + 2n\pi i}\right) = \left(2n + \frac{1}{4}\right)\pi i$, $n \in \mathbb{Z}$.

$\log\left(e^{\frac{5}{4}\pi i}\right) = \log\left(e^{\frac{5\pi}{4}i + 2n\pi i}\right) = \left(2n + \frac{5}{4}\right)\pi i = \left((2n+1) + \frac{1}{4}\right)\pi i$, $n \in \mathbb{Z}$.

Hence, $\log\left(i^{\frac{1}{2}}\right) = \left(n + \frac{1}{4}\right)\pi i$, $n \in \mathbb{Z}$.

(b) $\frac{1}{2}\log i = \frac{1}{2}\log e^{\frac{\pi}{2}i + 2n\pi i}$

$= \frac{1}{2}\left(\frac{1}{2} + 2n\right)\pi i$, $n \in \mathbb{Z}$

$= \left(\frac{1}{4} + n\right)\pi i$, $n \in \mathbb{Z}$

$= \log\left(i^{\frac{1}{2}}\right)$ by (a).

$\square$

P 103

2. (a) $(-i)^i = \left(e^{-\frac{\pi}{2}i}\right)^i$

$= e^{\frac{\pi}{2}}$

(b) $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$

$= \left(e \cdot e^{-\frac{2}{3}\pi i}\right)^{3\pi i}$

$= e^{(1-\frac{2}{3}\pi i)(3\pi i)}$

$= e^{3\pi i + 2\pi^2}$

$= -e^{2\pi^2}$

(c) $(1-i)^{4i} = \left(\sqrt{2}\, e^{-\frac{\pi}{4}i}\right)^{4i}$

$= \left(e^{\frac{1}{2}\ln 2 - \frac{\pi}{4}i}\right)^{4i}$

$= e^{\pi + i 2\ln 2}$

$= e^{\pi}\left[\cos(2\ln 2) + i\sin(2\ln 2)\right]$

$\square$